Math 565: Functional Analysis
Lecture 7

Complex vs. real linear functionals. Let X be a vector space over $\mathbb{C}$.

(a) If f is a complex linear functional on X then $\operatorname{Im} f(x) = -\operatorname{Re} f(ix)$ so $f(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix)$.

(b) Conversely, if u is a real linear functional on X, then $f(x) := u(x) - i u(ix)$ is a complex linear functional on X.

If, moreover, X is normed, then $\|u\| = \|\operatorname{Re} f\| = \|f\|$.

Proof. (a) If f is a complex lin. functional, then both $\operatorname{Re} f$ and $\operatorname{Im} f$ are real linear functionals, and $\operatorname{Im} f(x) = -\operatorname{Re} f(ix)$.

(b) If u is a real lin. functional on X then $f(x) := u(x) - i u(ix)$, then f is linear over $\mathbb{R}$ by definition, and also $f(ix) = u(ix) - i u(-x) = u(ix) + i u(x) = i(u(x) - i u(ix)) = i f(x)$, so f is also complex linear.

Finally, suppose X is a normed vector space. Since $|\operatorname{Re} f(x)| \leq |f(x)| \leq \|f\| \|x\|$, so $\|\operatorname{Re} f\| \leq \|f\|$. On the other hand, if $f(x) \neq 0$, then taking $\alpha := \overline{\operatorname{sgn} f(x)}$, we get $|f(x)| = \alpha \cdot f(x) = f(\alpha x) = |\operatorname{Re}(\alpha x)| \leq \|\operatorname{Re} f\| \|\alpha x\| = \|\operatorname{Re} f\| \|x\|$, so $\|f\| \leq \|\operatorname{Re} f\|$. $\boxed{\text{QED}}$

Dual of L^p.

Theorem. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $1 \leq p < \infty$. Then $L^p(\mu)^* \cong L^q(\mu)$ where q is the conj. exp. of p. More precisely, the map $L^q(\mu) \to L^p(\mu)^*$ by $g \mapsto I_g$ is an isometric isomorphism.

Proof. We have already shown that $g \mapsto I_g$ is an isometry, so it remains to show surjectivity. Fix a bdd linear functional $I \in L^p(\mu)^*$.

Case $\mu < \infty$. Then all simple functions are in $L^p(\mu)$ so for any μ-measurable set $B \subseteq X$, we can set $\rho(B) := I(\mathbb{1}_B)$. We show that ρ is a complex measure. Indeed,

$\rho(\emptyset) = \rho(\mathbb{1}_\emptyset) = I(0) = 0$ and if $B = \bigsqcup_{n \in \mathbb{N}} B_n$ then $\mathbb{1}_{\bigsqcup_{n \leq N} B_n} = \sum_{n \leq N} \mathbb{1}_{B_n} \nearrow \mathbb{1}_B$, so

$\| \mathbb{1}_B - \sum_{n \leq N} \mathbb{1}_{B_n} \|_p = \| \sum_{n > N} \mathbb{1}_{B_n} \|_p \to 0$ by DCT since $\mathbb{1}_B \in L^p(\mu)$ as $\mu < \infty$ and $p < \infty$ (otherwise $\| \sum_{n > N} \mathbb{1}_{B_n} \|_\infty = 1$). Since I is continuous, we get $I(\sum_{n \leq N} \mathbb{1}_{B_n}) \to I(\mathbb{1}_B)$ as $N \to \infty$, and by linearity, we have $\sum_{n \leq N} \rho(B_n) \to \rho(B)$ as $N \to \infty$, so $\sum_{n \in \mathbb{N}} \rho(B_n)$ converges to $\rho(B)$. Since regardless of rearrangement, the series converges to the same $\rho(B)$, by the Reimann rearrangement theorem, $\sum_{n \in \mathbb{N}} \rho(B_n)$ converges absolutely, i.e. $\sum_{n \in \mathbb{N}} |\rho(B_n)| < \infty$. Hence ρ is a complex measure. Observe that $\rho << \mu$ since if $\mu(B) = 0$ then $\mathbb{1}_B = 0$ a.e. so $\rho(B) = I(\mathbb{1}_B) = I(0) = 0$. Let $g := d\rho / d\mu$, so for each indicator function $\mathbb{1}_B$,

$$I(\mathbb{1}_B) = \rho(B) = \int_B g \, d\mu = \int \mathbb{1}_B g \, d\mu,$$

hence this is true for all simple functions instead of $\mathbb{1}_B$, by linearity. Thus: for all simple function $s : X \to \mathbb{C}$,

$$\left| \int s g \, d\mu \right| = |I(s)| \leq \|I\| \cdot \|s\|_p.$$

By the expression of q-norm via integrating against simple functs in L^p, we get $\|g\|_q \leq \|I\| < \infty$. Thus, I_g is a bdd linear functional on L^p and so is I, and they coincide on simple functions. But the set of simple functions is dense in L^p and I_g and I are continuous, so they must coincide on all of L^p, i.e. $I = I_g$.

Lastly note that such a g is unique, for example, because if there were another $\tilde{g} \in L^q$ with $I_g = I_{\tilde{g}}$, then $I_{g - \tilde{g}} = I_g - I_{\tilde{g}} = 0$, so $\|g - \tilde{g}\|_q = 0$ since the map $g - \tilde{g} \mapsto I_{g - \tilde{g}}$ is an isometry.

Case μ σ-finite. Let $X = \Uparrow_{n \in \mathbb{N}} X_n$ where $\mu(X_n) < \infty$. For each $n \in \mathbb{N}$, there is a unique (mod null) $g_n \in L^q(X_n, \mu|X_n)$ s.t. $I|_{L^p(X_n)} = I_{g_n}$. Naturally, $L^p(X_n) \subseteq L^p(X_{n+1}) \subseteq \dots L^p(X)$ by sending a function on X_n to a function defined as 0 outside of X_n and same on X_n. Thus, $L^p(X) = \Uparrow_{n \in \mathbb{N}} L^p(X_n)$ and by uniqueness, $g_{n+1}|_{X_n} = g_n$ a.e. so the pointwise limit $\lim_{n \to \infty} g_n =: g$ exists. Note that $|g_n|^q \nearrow |g|^q$ so MCT, $\|g\|_q = \lim_{n \to \infty} \|g_n\|_q \leq \|I\| < \infty$ so $g \in L^q(X)$. Finally, for each $f \in L^p(X)$, we have:

$$I_g(f) = \int fg \, d\mu \overset{DCT}{=} \lim_{n\to\infty} \int \mathbb{1}_{X_n} fg \, d\mu = \lim_{n\to\infty} \int \mathbb{1}_{X_n} f \cdot g_n \, d\mu = \lim_{n\to\infty} I_{g_n}(\mathbb{1}_{X_n} f) = \lim_{n\to\infty} I(\mathbb{1}_{X_n} f)$$

$= I(f)$ by the continuity of I since $\mathbb{1}_{X_n} f \to_{L^p} f$ again by DCT. QED

Remark. This theorem actually holds for all measures when $p > 1$ (i.e. $q < \infty$) by a simple measure exhaustion argument with σ-finite subsets of X, which works because for every $f \in L^p(X, \mu)$, the set $\{f \neq 0\}$ is σ-finite.

Riesz representation: the dual of $C_0(X)$.

As we saw, if $p = \infty$, then identifying $L^\infty(X, \mathcal{B}, \mu)$ with bdd $\mathcal{B}$-measurable functions on X, the proof of the map $L^1 \to (L^\infty)^*$ only gives that each $I \in (L^\infty)^*$ defines a finitely additive complex measure ρ since the continuity doesn't boost finite additivity to ctbl additivity. Indeed, $B(X, \mathcal{B})^*$ contains all finitely additive complex measures, including all ultrafilters on $\mathcal{B}$. We would like to shrink $B(X, \mathcal{B})^*$ by shrinking $B(X, \mathcal{B})$ to a very small closed subspace $V \subseteq B(X, \mathcal{B})$ such that V^* only consists of ctbly additive complex measures, so a finitely additive measure on $(X, \mathcal{B})$ integrated against functions in V behaves the same way as a countably additive measure. In other words, we need to boost finite additivity to ctbl additivity.

But we did this already when defining the Bernoulli measures on $2^{\mathbb{N}}$ and the Lebesgue measure on $\mathbb{R}^d$; see lectures 3 and 4 of Math 564, Fall 2025. Indeed, to prove that a measure on an algebra (of cylinders and of boxes, respectively) we defined was ctbly additive, we used compactness! This is what reduces infinite covers to finite, thus amplifying finite to ctbl additivity.

So we let X be a topological space and consider the space $C_c(X)$ of compactly supported (i.e. $\text{supp}(f) := \overline{\{f \neq 0\}}$ is compact) continuous functions on X. This is a nonclosed subspace of $BC(X)$ so we take its closure as V. To ensure the richness of $C_c(X)$, we need to assume that X is locally compact Hausdorff (lcH).